

Canonical Electromagnetic Observables for Systematic Characterization of Electric and Magnetic Wave Field Data on board Spacecraft

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Abstract. We present a new characterization of partially coherent electric and magnetic wave vector fields. This characterization is based on the 36 auto/cross correlations of the 3 + 3 complex Cartesian components of the electric and magnetic wave fields and is particularly suited for analyzing electromagnetic wave data on board spacecraft. Data from spacecraft based electromagnetic wave instruments are usually processed as data arrays. These data arrays however do not have a physical interpretation in themselves; they are simply a convenient storage format. In contrast, the characterization proposed here contains exactly the same information but are in the form of manifestly covariant space-time tensors. We call this data format the Canonical Electromagnetic Observables (CEO) since they correspond to unique physical observables. Some of them are already known, such as energy density, Poynting flux, stress tensor, etc, while others should be relevant in future space research. As an example we use this formalism to analyze data from a chorus emission in the mid-latitude magnetosphere, as recorded by the STAFF-SA instrument on board the Cluster-II spacecraft.

density matrix (in SI units) can be written

$$\epsilon_0 \begin{pmatrix} |E_x|^2 & E_x E_y^* & E_x E_z^* & cE_x B_x^* & cE_x B_y^* & cE_x B_z^* \\ E_y E_x^* & |E_y|^2 & E_y E_z^* & cE_y B_x^* & cE_y B_y^* & cE_y B_z^* \\ E_z E_x^* & E_z E_y^* & |E_z|^2 & cE_z B_x^* & cE_z B_y^* & cE_z B_z^* \\ cB_x E_x^* & cB_x E_y^* & cB_x E_z^* & |cB_x|^2 & c^2 B_x B_y^* & c^2 B_x B_z^* \\ cB_y E_x^* & cB_y E_y^* & cB_y E_z^* & c^2 B_y B_x^* & |cB_y|^2 & c^2 B_y B_z^* \\ cB_z E_x^* & cB_z E_y^* & cB_z E_z^* & c^2 B_z B_x^* & c^2 B_z B_y^* & |cB_z|^2 \end{pmatrix}$$

This electromagnetic (EM) sixtor matrix has in various guises, such as wave-distribution functions (WDF) (Storey and Lefeuvre, 1974) and so on, been useful in the analysis of EM vector field data from spacecraft, for instance on the Cluster and Polar missions. The second order coherency matrix is important from a statistical viewpoint since it completely describes a wide sense stationary vector signal. From a physical point of view it is important since energy density, EM wave polarization, and similar quantities can be derived from its components.

The EM sixtor matrix can be seen as a generalization of the coherency matrix in optics, which is usually a 2×2 Hermitian matrix, describing the transverse field. The coherency matrix description is convenient as a data storage format but in practice, it is more common to instead use the four Stokes parameters, as they are physically more intuitive.

One may ask what the Stokes description for the full EM field would be? One way is to decompose the 6×6 coherency matrix in terms of a complete basis set of unitary matrices, as per Samson and Olsen (1980). The problem with this approach is that such a decomposition is not unique; there is an infinite number of unitary bases.

In this paper we introduce a unique set of parameters, analogous to the Stokes parameters, but generalized to the full electric and magnetic wave fields. We call these parameters the Canonical Electromagnetic Observables (CEO), due to their uniqueness, which comes from the fact that they are irreducible under Lorentz transformations. Furthermore, they

1 Introduction

When analyzing time varying electric and magnetic vector field data from spacecraft, it is common to construct a 6×6 matrix from the complex vectors $(\mathbf{E}, c\mathbf{B})$, sometimes jointly called a sixtor. In a Cartesian coordinate system, this energy

are not merely “parameters”, they are proper space-time tensors.

Some examples of the CEO in vacuum are the EM energy density $\epsilon_0(|\mathbf{E}|^2 + |\mathbf{B}|^2)/2$, and the energy flux density (proportional to the Poynting vector) $\text{Re}[\mathbf{E} \times \mathbf{B}^*]/Z_0$, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the vacuum impedance. These quantities can be quite easily identified from the sixtor matrix by inspection, for instance its trace is the energy density. Other CEO are not so easily identified.

The procedure to calculate the fundamental CEO, in terms of six real irreducible space-time 4-tensors, as well as the corresponding three-dimensional representation will be given in Section 2. A two-dimensional CEO representation is given in Section 3.

1.1 Covariance in space physics

Perhaps even more important for space borne observations: the EM sixtor matrix is not covariant according to the requirements of special relativity. Spacecraft are constantly moving and often spinning observation platforms. EM wave measurements become Doppler shifted and data must often be “despun”. For the four 3×3 sub-matrices: $\epsilon_0 \mathbf{E} \otimes \mathbf{E}^*$, $\mathbf{E} \otimes \mathbf{B}^*/Z_0$, $\mathbf{B} \otimes \mathbf{E}^*/Z_0$, and $\mathbf{B} \otimes \mathbf{B}^*/\mu_0$, where \otimes denote the outer (tensor) product, despining is straight forward by applying rotation matrices \mathbf{R} from left and \mathbf{R}^T from right, *e.g.* $\mathbf{E}' \otimes \mathbf{E}' = \mathbf{R} \mathbf{E} \otimes \mathbf{E}^T \mathbf{R}^T$. To rotate the full EM sixtor matrix, similar operations must be performed four times. This is awkward and the resulting 6×6 matrix is still not covariant. For EM wave measurements in space plasma the last remark can be crucial.

A Lorentz boost is the translation from one Lorentz frame to another one moving at velocity \mathbf{v} . A Lorentz boost does not necessarily imply relativistic speeds, which is a common misconception; and therefore it do not by itself preclude what is typically associated with relativistic effects. It is simply a quite general recipe to make two different observers agree on a physical observation. The Lorentz boost of the EM field vectors can be written $\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and $\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2)$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. As a matter of fact, the Lorentz boost is the essence of the well-known frozen-in field line theorem¹ from magnetohydrodynamics (MHD); a theory which is commonly used to model the solar wind plasma. In a plasma, relativity comes into play at very a fundamental level since the electromagnetic (Lorentz) force dominates the vast majority of all plasma interactions.

Another example illustrates the problem to separate time (frequency) and space (wave vector) in EM wave observations on board a spacecraft. Assume that we observe a wave mode which is described by an angular frequency ω and wave vector \mathbf{k} . We can write this as a 4-vector $(\omega, c\mathbf{k})$. Let's

make a Lorentz boost in the \mathbf{v} direction:

$$\omega' = \gamma(\omega - \mathbf{k} \cdot \mathbf{v}) \quad (1)$$

$$c\mathbf{k}' = c\mathbf{k} + \left[\frac{\gamma - 1}{v^2} (c\mathbf{k} \cdot \mathbf{v}) - \gamma\omega \right] \frac{\mathbf{v}}{c} \quad (2)$$

What happens now for a stationary (DC) field structure moving with the solar wind plasma? We then have $\omega = 0$ and $|c\mathbf{k}| \neq 0$. For a satellite moving with velocity \mathbf{v} relative to the DC field structure, it is justified to set $\gamma \approx 1$ (the solar wind speed seldom reaches more than 900 km/s and using this value we obtain $\gamma \approx 1.0000045 \gtrsim 1$); Eqs. (1) and (2) are then reduced to

$$\omega' \approx -\mathbf{k} \cdot \mathbf{v} \quad (3)$$

$$c\mathbf{k}' \approx c\mathbf{k} \quad (4)$$

We can see that the DC field structure is not Lorentz contracted appreciably at this low velocity, $\mathbf{k}' \approx \mathbf{k}$. However, there is a dramatic change in the observed frequency, which for a head-on encounter with the structure is registered as $\omega' \approx kv$ rather than zero. The observed frequency is proportional to the dimension of the structure, which we take to be in the order of one wavelength, $\lambda = 2\pi/k$. Taking $v = 900$ km/s a 900 km DC field structure would now register as 1 Hz, a 90 km structure as 10 Hz, and a 9 km structure as 100 Hz, etc.

These simple examples clearly show that a space-time (co-)variant description is necessary even if $\gamma \gtrsim 1$. The frequency (time) and the associated wave vector (space) can not be treated separately but must be considered together, as a space-time 4-tensor.

The Maxwell equations are inherently relativistic and can easily be put into a covariant form using 4-tensors. From a theoretical point of view, this fact alone provides a very good argument why one should try to express also the second order properties of the EM fields using a covariant formalism. This was recently carried out by the authors and published in a recent paper Carozzi and Bergman (2006). In this paper we introduced a complete set of space-time tensors, which can fully describe the second order properties of EM waves. We call this set of tensors the Canonical Electromagnetic Observables (CEO); in analogy with Wolf's analysis of the Stokes parameters Wolf (1954). We suggest that the CEO could be used as an alternative to the EM sixtor matrix. Not only are the CEO covariant, but they are all real valued and provide a useful decomposition of the sixtor matrix into convenient physical quantities, especially in the three-dimensional (3D), so-called scalar-vector-tensor (SVT) classification; see section 2.2. The CEO have all dimension energy density but have various physical interpretations as will be discussed in what follows.

¹If $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ in a plasma, the magnetic field lines change as though they are convected with velocity \mathbf{v} , *i.e.*, they are frozen to the plasma flow. This is the frozen-in field line theorem of ideal MHD.

CEO 4-tensor	Rank (Symmetry)	Proper + Pseudo –	Number of observables
C_+	0	+	1
C_-	0	–	1
T	2(S)	+	9
U	2(S)	–	9
Q	2(A)	–	6
W	4(M)	+	10

Table 1. CEO in space-time classification, *i.e.*, 4-tensor notation: $1 + 1 + 9 + 9 + 6 + 10 = 36$ observables.

2 Canonical Electromagnetic Observables

The CEO set was derived from the complex Maxwell field strength $F^{\alpha\beta}$. Other possibilities, such as using the 4-potential A^μ or using a spinor formalism Barut (1980), were considered but discarded due to their lack of physical content. The 4-potential is not directly measurable and it is furthermore gauge dependent. Spinor formalism has been proved possible to use Sundkvist (2006) but we believe the space-time tensor formalism to be more intuitive and convenient to use.

In the quantum theory of light, observables of an EM field are ultimately constructed from a complex field strength; see Wolf (1954). The simplest of these observables are sesquilinear-quadratic (Hermitian quadratic) in $F^{\alpha\beta}$, *i.e.*, they are functions of the components of $F^{\alpha\beta}\bar{F}^{\gamma\delta}$, which is a 4-tensor of rank four. Here we have chosen to denote the complex conjugate of the field strength with a bar over the field symbol in order not to confuse it with the dual field strength, which we denote by a superscript star to the left of the field symbol. We showed that it was possible to decompose $F^{\alpha\beta}\bar{F}^{\gamma\delta}$ into a unique set of tensors, the CEO, which are real irreducible under the full Lorentz group. We shall not repeat the derivation here but will instead discuss the space-time (4-tensor) and three-dimensional (3-tensor) representations of the CEO.

2.1 Fundamental space-time representation

In terms of the Maxwell field strength $F^{\alpha\beta}$, the CEO are organized in the six real irreducible 4-tensors C_+ , C_- , $Q^{\alpha\beta}$, $T^{\alpha\beta}$, $U^{\alpha\beta}$, and $W^{\alpha\beta\gamma\delta}$. This is the fundamental space-time representation of the CEO; their properties are listed in Table 2.1.

The CEO 4-tensors are defined as follows: the two scalars are the vacuum proper- and pseudo-Lagrangians,

$$C_+ := (\bar{F}_{\alpha\beta}F^{\alpha\beta} - {}^*F_{\alpha\beta}{}^*F^{\alpha\beta})/2, \quad (5)$$

$$C_- := (\bar{F}_{\alpha\beta}{}^*F^{\alpha\beta} + {}^*F_{\alpha\beta}F^{\alpha\beta})/2, \quad (6)$$

respectively, where we have used the dual of $F^{\alpha\beta}$ defined as

$${}^*F^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\gamma\delta}F^{\gamma\delta}. \quad (7)$$

The three second rank tensors consist of the two symmetric tensors

$$T^{\alpha\beta} := (\bar{F}^\alpha{}_\mu F^{\mu\beta} + {}^*F^\alpha{}_\mu {}^*F^{\mu\beta})/2, \quad (8)$$

$$U^{\alpha\beta} := i(\bar{F}^\alpha{}_\mu {}^*F^{\mu\beta} - {}^*F^\alpha{}_\mu F^{\mu\beta})/2, \quad (9)$$

and the antisymmetric tensor

$$Q^{\alpha\beta} := i(\bar{F}^\alpha{}_\mu F^{\mu\beta} - {}^*F^\alpha{}_\mu {}^*F^{\mu\beta} - 2C_+\eta^{\alpha\beta})/2. \quad (10)$$

The symmetric second rank tensor $T^{\alpha\beta}$ is the well-known EM energy-stress tensor, which contains the total energy, flux (Poynting vector), and stress (Maxwell stress tensor) densities. The other two second rank tensors, $U^{\alpha\beta}$ and $Q^{\alpha\beta}$, respectively, are less well-known. The symmetric $U^{\alpha\beta}$ tensor is similar to $T^{\alpha\beta}$ in that it contains active energy densities but in $U^{\alpha\beta}$ these densities are weighted and depend on the handedness (spin, helicity, polarization, chirality) of the EM field. Therefore, we have chosen to call them “handed” energy densities. The anti-symmetric tensor $Q^{\alpha\beta}$ on the other hand is very different in that it only contains reactive energy densities, which are both total (imaginary part of the complex Poynting vector) and handed.

The fourth rank tensor is

$$W^{\alpha\beta\gamma\delta} := (\bar{F}^{\alpha\beta}F^{\gamma\delta} - {}^*F^{\alpha\beta}{}^*F^{\gamma\delta})/2 - 2iQ^{[\alpha\delta}\eta^{\gamma\beta]} - \frac{2}{3}C_+\eta^{\alpha[\delta}\eta^{\gamma\beta]} - \frac{1}{3}C_-\epsilon^{\alpha\beta\gamma\delta} \quad (11)$$

where the square brackets denotes antisymmetrization over the enclosed indices, *e.g.*, $T^{\alpha[\delta}g^{\gamma]\beta} = \frac{1}{2}(T^{\alpha\delta}g^{\gamma\beta} - T^{\alpha\gamma}g^{\delta\beta})$, and nested brackets are not operated on by enclosing brackets, *e.g.*, $T^{[\alpha[\delta}g^{\gamma]\beta]} = \frac{1}{4}(T^{\alpha\delta}g^{\gamma\beta} - T^{\alpha\gamma}g^{\delta\beta} - T^{\beta\delta}g^{\gamma\alpha} + T^{\beta\gamma}g^{\delta\alpha})$. It fulfills the symmetries $W^{\alpha\beta\gamma\delta} = W^{\beta\alpha\gamma\delta} = W^{\alpha\beta\delta\gamma} = W^{\gamma\delta\alpha\beta}$ and $W^{\alpha[\beta\gamma\delta]} = 0$.

This real irreducible rank four tensor, Eq. (11), was discovered by us² and published in Carozzi and Bergman (2006), and is still under investigation; it is an extremely interesting geometrical object, having a structure identical to the Weyl tensor in general relativity; see Weinberg (1972). We have found that it contains a four-dimensional generalization of the Stokes parameters, as will be demonstrated in section 3 for the two-dimensional (2D) case. It contains both reactive total and reactive handed energy densities.

2.2 Three-dimensional representation

The fundamental space-time 4-tensor CEO can be written in terms of the three-dimensional \mathbf{E} and \mathbf{B} vectors, *i.e.*, 3-tensors. This is convenient because it allows us to use intuitive physical quantities. To systematize the 3D representation of the CEO, we will use a physical classification

²To the best of our knowledge, the $W^{\alpha\beta\gamma\delta}$ tensor has never before been published in the literature.

	Scalars	Vectors	Tensors
Active Total (AT)	$u \{S_{AT}\}$	$\mathbf{P} \{\mathbf{V}_{AT}\}$	$\mathbf{T} \{\mathbf{T}_{AT}\}$
Active Handed (AH)	$w \{S_{AH}\}$	$\mathbf{V} \{\mathbf{V}_{AH}\}$	$\mathbf{U} \{\mathbf{T}_{AH}\}$
Reactive Total (RT)	$l \{S_{RT}\}$	$\mathbf{R} \{\mathbf{V}_{RT}\}$	$\mathbf{X} \{\mathbf{T}_{RT}\}$
Reactive Handed (RH)	$a \{S_{RH}\}$	$\mathbf{O} \{\mathbf{V}_{RH}\}$	$\mathbf{Y} \{\mathbf{T}_{RH}\}$

Table 2. CEO in scalar-vector-tensor (SVT) classification, *i.e.*, 3-tensor notation: $4 \times (1 + 3 + 5) = 36$ observables. In curly brackets we provide an alternative notation of the CEO.

where we organize the CEO into four groups, which have been introduced briefly in the previous section: the *active total*, *active handed*, *reactive total*, and *reactive handed* CEO parameter groups, respectively. In addition, we will use a coordinate-free 3D formalism and classify the CEO parameters according to rank, *i.e.*, as scalars, 3-vectors, and rank two 3-tensors (SVT classification). The 3D CEO are listed in Table 2.2. The alternative SVT nomenclature, enclosed by curly brackets in the table, was proposed in Olofsson (2001), which used S , \mathbf{V} and \mathbf{T} , to denote all the scalars, vectors, and tensors, respectively. The subscript indexing scheme used here is different to that used by Olofsson. The CEO 3-tensors are defined as follows.

The “total” parameters are:

$$u = \varepsilon_0 T^{00} = \varepsilon_0 (|\mathbf{E}|^2 + |\mathbf{B}|^2) / 2 \quad (12)$$

$$\mathbf{P} = \varepsilon_0 T^{i0} = \text{Re} [\mathbf{E} \times \mathbf{B}^*] / Z_0 \quad (13)$$

$$\mathbf{T} = \varepsilon_0 T^{ij} = u \mathbf{1}_3 - \varepsilon_0 \text{Re} [\mathbf{E} \otimes \mathbf{E}^* + c^2 \mathbf{B} \otimes \mathbf{B}^*] \quad (14)$$

$$(15)$$

where $\mathbf{1}_3$ is the identity matrix in three dimensions. This is the 3D representation of the well-known energy-stress 4-tensor $\varepsilon_0 T^{\alpha\beta}$, defined by Eq. (8).

The “handed” parameters are:

$$w = \varepsilon_0 U^{00} = \text{Im} [\mathbf{E} \cdot \mathbf{B}^*] / Z_0 \quad (16)$$

$$\mathbf{V} = \varepsilon_0 U^{i0} = -\varepsilon_0 \text{Im} [(\mathbf{E} \times \mathbf{E}^* + c^2 \mathbf{B} \times \mathbf{B}^*)] / 2 \quad (17)$$

$$\mathbf{U} = \varepsilon_0 U^{ij} = v \mathbf{1}_3 - \text{Im} [\mathbf{E} \otimes \mathbf{B}^* - \mathbf{B} \otimes \mathbf{E}^*] / Z_0 \quad (18)$$

This is the 3D representation of the handed energy-stress 4-tensor $\varepsilon_0 U^{\alpha\beta}$, defined by Eq. (9).

The “reactive total” parameters are:

$$l = \varepsilon_0 C_+ = \varepsilon_0 (|\mathbf{E}|^2 - |\mathbf{B}|^2) / 2 \quad (19)$$

$$\mathbf{R} = \varepsilon_0 Q^{i0} = -\text{Im} [\mathbf{E} \times \mathbf{B}^*] / Z_0 \quad (20)$$

$$\mathbf{X} = \varepsilon_0 W^{i0j0} = \frac{1}{2} \left(\varepsilon_0 \text{Re} [\mathbf{E} \otimes \mathbf{E}^* - c^2 \mathbf{B} \otimes \mathbf{B}^*] - \frac{2}{3} l \mathbf{1}_3 \right) \quad (21)$$

Contrary to the active, total and handed, parameter groups above, the reactive total parameter group have no single corresponding 4-tensor. Instead it is composed of parts from three different CEO space-time tensors: the vacuum proper-Lagrangian defined by Eq. (5), the reactive energy flux density from Eq. (10), and the generalized Stokes parameters

corresponding to the auto-correlated \mathbf{E} and \mathbf{B} fields from Eq. (11).

The “reactive handed” parameters are:

$$a = \varepsilon_0 C_- = -\text{Re} [\mathbf{E} \cdot \mathbf{B}^*] / Z_0 \quad (22)$$

$$\mathbf{O} = \varepsilon_0 \frac{1}{2} \varepsilon_{kl}^j Q^{kl} = -\frac{1}{2} \text{Im} [(\mathbf{E} \times \mathbf{E}^* - c^2 \mathbf{B} \times \mathbf{B}^*)] \quad (23)$$

$$\mathbf{Y} = \varepsilon_0 \frac{1}{2} \varepsilon_{kl}^j W^{i0kl} = \frac{1}{2} \left(\text{Re} [\mathbf{E} \otimes \mathbf{B}^* + \mathbf{B} \otimes \mathbf{E}^*] / Z_0 - \frac{2}{3} a \mathbf{1}_3 \right) \quad (24)$$

Also for this parameter group, there is no single corresponding 4-tensor. The reactive handed group is composed of parts from three CEO space-time tensors: the vacuum pseudo-Lagrangian, defined by Eq. (6), the reactive handed energy flux density from Eq. (10), and the generalized Stokes parameters corresponding to the cross-correlated \mathbf{E} and \mathbf{B} fields from Eq. (11).

3 CEO in two dimensions

Up until now we have assumed that all three Cartesian components of both the electric field, \mathbf{E} , and the magnetic field, \mathbf{B} , are measured. One may ask what happens if some components are not measured; can all the 36 parameters of the CEO be retained? Of course this is not possible, some information is certainly lost in this case, but what one can do is to construct a set of parameters analogous to CEO in two-dimensions. As will be shown, a total of $4 \times (1 + 1 + 2) = 16$ CEO 2D parameters can be derived.

Assume that we can measure the electric field and the magnetic field in a plane which we can say is the xy -plane without loss of generality. Let the two-dimensional (2D) fields in this plane be denoted $\mathbf{E}_{2D} := (E_x, E_y)$ and $\mathbf{B}_{2D} := (B_x, B_y)$, and define the scalar product between 2D vectors as

$$\mathbf{E}_{2D} \cdot \mathbf{B}_{2D}^* = E_x B_x^* + E_y B_y^* \quad (25)$$

and the cross product as

$$\mathbf{E}_{2D} \times \mathbf{B}_{2D}^* = E_x B_y^* - E_y B_x^* \quad (26)$$

and the outer (tensor) product as

$$\mathbf{E}_{2D} \otimes \mathbf{B}_{2D}^* = \begin{pmatrix} E_x B_x^* & E_x B_y^* \\ E_y B_x^* & E_y B_y^* \end{pmatrix} \quad (27)$$

We will however not need to consider all the components of the 2D direct product since the 2-tensors we will consider are all symmetric and traceless. Hence, we only want the parameters which correspond to Pauli spin matrix components

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (28)$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (29)$$

The Pauli components can be extracted from a 2D matrix by matrix multiplying by a Pauli spin matrix and then taking the trace (Tr), that is

$$\begin{aligned} \text{Tr}\{(\mathbf{E}_{2D} \otimes \mathbf{B}_{2D}^*) \sigma_x\} &= (\mathbf{E}_{2D} \otimes \mathbf{B}_{2D}^*) \cdot \sigma_x \\ &:= E_x B_y^* + E_y B_x^* \end{aligned} \quad (30)$$

where the double dots, \cdot , have been introduced as a symbol for the double scalar product, see Lebedev and Cloud (2003).

We can derive a set of two-dimensional canonical electromagnetic parameters from the full CEO by formally taking

$$E_z \equiv B_z \equiv 0 \quad (31)$$

and discarding all the parameters that are identically zero. In this way we obtain the following set, which we write in the 2D formalism introduced above.

The “active total” 2D parameters are:

$$\begin{aligned} u_{2D} &= \varepsilon_0 \left(|\mathbf{E}_{2D}|^2 + c^2 |\mathbf{B}_{2D}|^2 \right) / 2 \\ &= \varepsilon_0 \left(|E_x|^2 + |E_y|^2 + |cB_x|^2 + |cB_y|^2 \right) / 2 \end{aligned} \quad (32)$$

$$\begin{aligned} P_z &= \text{Re} [\mathbf{E}_{2D} \times \mathbf{B}_{2D}^*] / Z_0 \\ &= \text{Re} [E_x B_y^* - E_y B_x^*] / Z_0 \end{aligned} \quad (33)$$

$$\begin{aligned} T_{\sigma_z} &= \varepsilon_0 \text{Re} [\mathbf{E}_{2D} \otimes \mathbf{E}_{2D}^* + c^2 \mathbf{B}_{2D} \otimes \mathbf{B}_{2D}^*] \cdot \sigma_z / 2 \\ &= \varepsilon_0 \left(|E_x|^2 - |E_y|^2 + |cB_x|^2 - |cB_y|^2 \right) / 2 \end{aligned} \quad (34)$$

$$\begin{aligned} T_{\sigma_x} &= \varepsilon_0 \text{Re} [\mathbf{E}_{2D} \otimes \mathbf{E}_{2D}^* + c^2 \mathbf{B}_{2D} \otimes \mathbf{B}_{2D}^*] \cdot \sigma_x / 2 \\ &= \varepsilon_0 \text{Re} [E_x E_y^* + B_x B_y^*] \end{aligned} \quad (35)$$

The “active handed” 2D parameters are:

$$\begin{aligned} w_{2D} &= \text{Im} [\mathbf{E}_{2D} \cdot \mathbf{B}_{2D}^*] / Z_0 \\ &= \text{Im} [E_x B_x^* + E_y B_y^*] / Z_0 \end{aligned} \quad (36)$$

$$\begin{aligned} V_z &= \varepsilon_0 \text{Im} [\mathbf{E}_{2D} \times \mathbf{E}_{2D}^* + c^2 \mathbf{B}_{2D} \times \mathbf{B}_{2D}^*] / 2 \\ &= \varepsilon_0 \text{Im} [E_x E_y^* + c^2 B_x B_y^*] / 2 \end{aligned} \quad (37)$$

$$\begin{aligned} U_{\sigma_z} &= \text{Im} [\mathbf{E}_{2D} \otimes \mathbf{B}_{2D}^* - \mathbf{B}_{2D} \otimes \mathbf{E}_{2D}^*] \cdot \sigma_z / 2Z_0 \\ &= \text{Im} [E_x B_x^* - E_y B_y^*] / Z_0 \end{aligned} \quad (38)$$

$$\begin{aligned} U_{\sigma_x} &= \text{Im} [\mathbf{E}_{2D} \otimes \mathbf{B}_{2D}^* - \mathbf{B}_{2D} \otimes \mathbf{E}_{2D}^*] \cdot \sigma_x / 2Z_0 \\ &= \text{Im} [E_x B_y^* + E_y B_x^*] / Z_0 \end{aligned} \quad (39)$$

The “reactive total” 2D parameters are:

$$\begin{aligned} l_{2D} &= \varepsilon_0 \left(|\mathbf{E}_{2D}|^2 - c^2 |\mathbf{B}_{2D}|^2 \right) / 2 \\ &= \varepsilon_0 \left(|E_x|^2 + |E_y|^2 - |cB_x|^2 - |cB_y|^2 \right) / 2 \end{aligned} \quad (40)$$

$$\begin{aligned} R_z &= \text{Im} [\mathbf{E}_{2D} \times \mathbf{B}_{2D}^*] / Z_0 \\ &= \text{Im} [E_x B_y^* - E_y B_x^*] / Z_0 \end{aligned} \quad (41)$$

$$\begin{aligned} X_{\sigma_z} &= \varepsilon_0 \text{Re} [\mathbf{E}_{2D} \otimes \mathbf{E}_{2D}^* - c^2 \mathbf{B}_{2D} \otimes \mathbf{B}_{2D}^*] \cdot \sigma_z / 2 \\ &= \varepsilon_0 \left(|E_x|^2 - |E_y|^2 - |cB_x|^2 + |cB_y|^2 \right) / 2 \end{aligned} \quad (42)$$

$$\begin{aligned} X_{\sigma_x} &= \varepsilon_0 \text{Re} [\mathbf{E}_{2D} \otimes \mathbf{E}_{2D}^* - \mathbf{B}_{2D} \otimes \mathbf{B}_{2D}^*] \cdot \sigma_x / 2 \\ &= \varepsilon_0 \text{Re} [E_x E_y^* - c^2 B_x B_y^*] \end{aligned} \quad (43)$$

The “reactive handed” 2D parameters are:

$$\begin{aligned} a_{2D} &= \text{Re} [\mathbf{E}_{2D} \cdot \mathbf{B}_{2D}^*] / Z_0 \\ &= \text{Re} [E_x B_x^* + E_y B_y^*] / Z_0 \end{aligned} \quad (44)$$

$$\begin{aligned} O_z &= \varepsilon_0 \text{Im} [\mathbf{E}_{2D} \times \mathbf{E}_{2D}^* - c^2 \mathbf{B}_{2D} \times \mathbf{B}_{2D}^*] \\ &= \varepsilon_0 \text{Im} [E_x E_y^* - c^2 B_x B_y^*] \end{aligned} \quad (45)$$

$$\begin{aligned} Y_{\sigma_z} &= \text{Re} [\mathbf{E}_{2D} \otimes \mathbf{B}_{2D}^* + \mathbf{B}_{2D} \otimes \mathbf{E}_{2D}^*] \cdot \sigma_z / 2Z_0 \\ &= \text{Re} [E_x B_x^* - E_y B_y^*] / Z_0 \end{aligned} \quad (46)$$

$$\begin{aligned} Y_{\sigma_x} &= \text{Re} [\mathbf{E}_{2D} \otimes \mathbf{B}_{2D}^* + \mathbf{B}_{2D} \otimes \mathbf{E}_{2D}^*] \cdot \sigma_x / 2Z_0 \\ &= \text{Re} [E_x B_y^* + E_y B_x^*] / Z_0 \end{aligned} \quad (47)$$

We can associate names with these parameters as listed in Table 3. The first four parameters, which we call the “total” 2D CEO parameters are all well known. These parameters are also known by different names, e.g., the total energy flux is also known as the Poynting vector (z -component), and the total energy stress is known as the Maxwell stress tensor (difference of diagonal components and off-diagonal component). The remaining three sets of 2D CEO parameters are less well known. We will not be able to provide a full physical interpretation of each of these parameters; indeed their role in space plasma physics is yet to be fully explored. We will only mention that the “handed” parameters involve spin (helicity, chirality, polarization) weighted energy, i.e., the energy of the right-hand wave modes are weighted positively and the energy of left-hand wave modes are weighted negatively, and these weighted energies are then added. Its flux corresponds to the concept of ellipticity and for the case of vacuum, it is numerically equivalent to Stokes V parameter. The reactive energy densities come in two groups: the “reactive total” and the “reactive handed” 2D CEO parameter groups. From the “reactive total” group, we now recognize the reactive energy flux density, as well as the EM

Symbol	Name
u_{2D}	Total energy
P_z	Total energy flux
T_{σ_z}	Total energy stress σ_z -component
T_{σ_x}	Total energy stress σ_x -component
w_{2D}	Handed energy
V_z	Handed energy flux
U_{σ_z}	Handed energy stress σ_z -component
U_{σ_x}	Handed energy stress σ_x -component
l_{2D}	Vacuum proper-Lagrangian
R_z	Reactive energy flux
X_{σ_z}	EM Stokes parameter Q auto-type
X_{σ_x}	EM Stokes parameter U auto-type
a_{2D}	Vacuum pseudo-Lagrangian
O_z	Reactive handed energy flux
Y_{σ_z}	EM Stokes parameter Q cross-type
Y_{σ_x}	EM Stokes parameter U cross-type

Table 3. Naming scheme for the 2D CEO parameters.

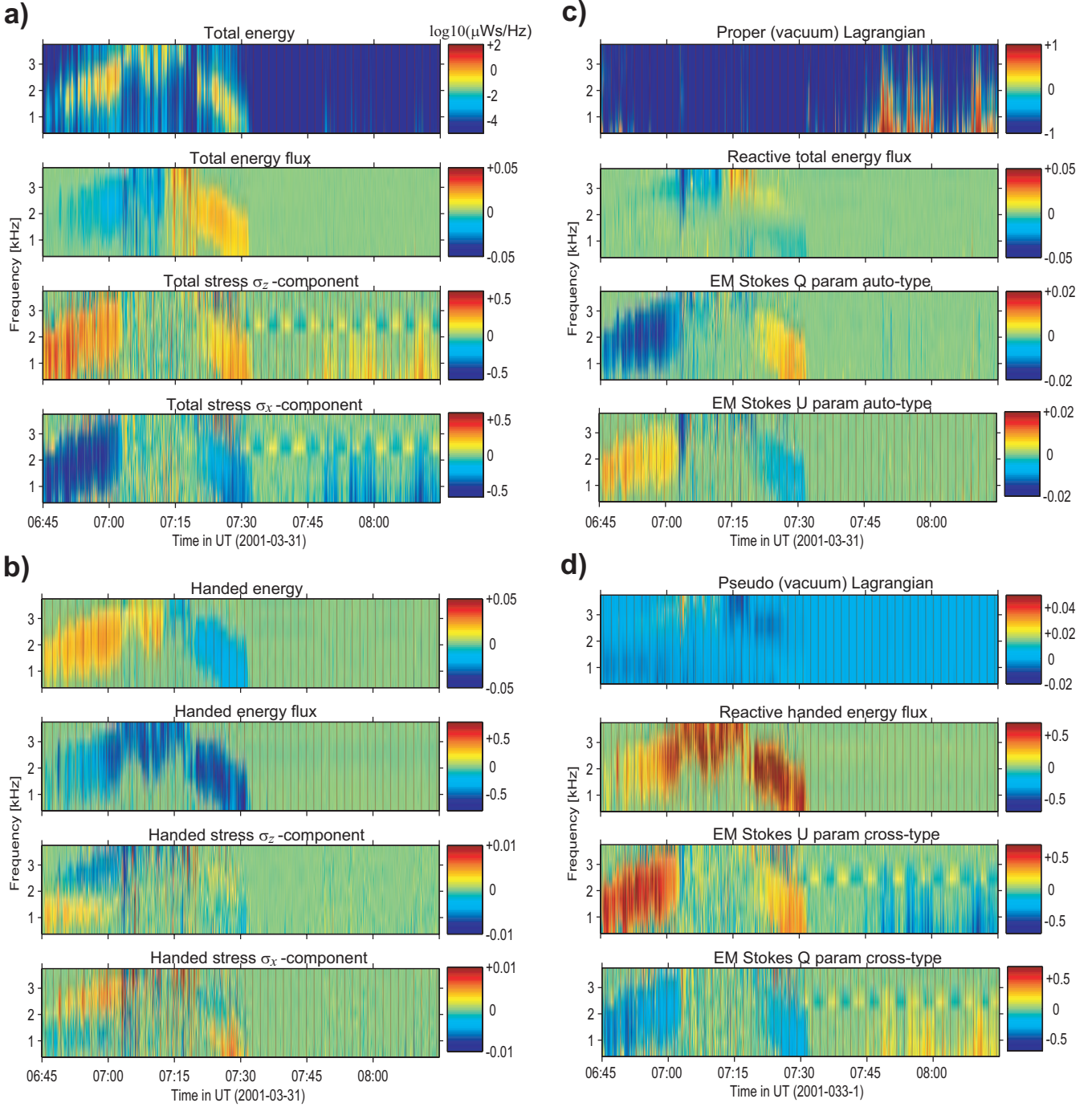


Fig. 1. Example dynamic spectra of the 16 normalized two-dimensional CEO parameters. The parameters were computed from STAFF-SA data from Cluster space-craft 2 using the ISDAT database system. The following normalization has been applied: each parameter has been divided by the total energy except the total energy itself. Thus all spectral values are in dimensionless unit except for the total energy. This Figure can be compared with Fig. 1 in Parrot et al. (2003). The 16 parameters are subdivided into a) the active total energy parameters, b) the active handed energy parameters, c) the reactive total energy parameters, and d) the reactive handed energy parameters. Note that all the quantities are purely electromagnetic in origin and so do not refer to contributions from the plasma.

Stokes Q and U parameters, which here are of the auto-type; the vacuum proper-Lagrangian needs no further introduction. The “reactive handed” group contain the handed counterparts of the reactive energy flux density and EM Stokes parameters, which here are of the cross-type; the vacuum pseudo-Lagrangian is well-known.

4 Application of CEO to Cluster data

Let us demonstrate that the CEO parameters can easily be computed from actual data. Assuming that we have measurements from a vector magnetometer and an electric field instrument, all that is required is to auto/cross-correlate all measured components and then form the appropriate linear combination introduced above.

As an example we will consider the STAFF-SA dataset on the Cluster-II space-craft mission Escoubet et al. (1997). The STAFF-SA instrument Cornilleau-Wehrin (1997) is well suited for the CEO parameters since it outputs auto/cross-correlation of electric and magnetic field components; however as Cluster does not measure one of the electric field components (namely the component normal to the spin-plane of the space-craft) we can only use the 2D version of the CEO introduced in the previous section.

For this particular example, we re-process the high-band part of STAFF-SA data from an event discussed in Parrot et al. (2003) from 2001-03-31 UT. In Fig. 1 of this paper, Parrot *et al* display certain parameters based on the STAFF-SA data computed using a numerical software package called PRASSADCO; see Santolík (2003); Santolík et al. (2006). The interesting feature of the 2D CEO parameters is that they are the complete set of electromagnetic field observables in the spin-plane of the space-craft; and indeed, they use up all the parameters in the STAFF-SA dataset except for the magnetic field in the spin direction. Each CEO is a distinct physical quantity and examination of the panels in Fig. 1 indicates that this is indeed the case, since besides showing a common chorus feature (the arch to the left in each panel) there are unique points in each of the panels.

Besides being a complete description of the electromagnetic observables, the fact that the CEO parameters are based on parameters that conform with the physics of space-time means that we can expect physical phenomenon to be measured properly. Seeing as how the CEO parameters have not been explicitly measured in the past, we can expect that their future use may lead to new physical insights, especially since several of the parameters are completely new to space-physics. As an example consider again the data shown in Fig 1. It is interesting to note that the reactive total energy flux is only significant close to the equator; this implies that the equator is the source region for the chorus events, since reactive energy flux is typically large close to radiating objects due to large standing energy fields. One can also see a modulation at 2.5 kHz in the EM Stokes parameters. If this

is a physical phenomenon it would be indicative of Faraday rotation. Also there seems to be frequency dispersion in the handed stress since its components changes sign with frequency. Finally, the handed energy clearly shows the handedness of the chorus emissions on its own, without recourse to the sign of the total energy flux.

5 Conclusions

The proposed CEO parameters conveniently organize the measurements of the full EM wave field. They are physically meaningful quantities, *i.e.* they transform as geometric (Minkowski space-time) objects and they are mathematically unique (they are irreducible tensors). The CEO retain all information, *i.e.* nothing is lost, and a linear transformation back to the full sixtor form exists. Through parameter subset selection they could enable considerable data reduction. These parameters have clear despinning properties and the scalar quantities do not even need despinning. Some of the CEO parameters have not been used before to describe EM wave fields and can thus be used to reveal new physical insights when applied to the analysis of EM wave field data measured by spacecraft. To this end a particularly useful decomposition of the 36 second order EM components into twelve 3-tensor quantities have been provided. All the CEO are real valued and unique. The active CEO can propagate to infinity. Notably, the tensors $T^{\alpha\beta}$ and $U^{\alpha\beta}$ obey the (vacuum) conservation laws $\partial_\alpha T^{\alpha\beta} = 0$ and $\partial_\alpha U^{\alpha\beta} = 0$, respectively, see Bergman et al. (2008). The reactive CEO on the other hand do not obey conservation laws, and hence, can not propagate to infinity. They are nevertheless important since they provide clues to investigate the properties of the near field. As illustrated in Fig.1, this could be very useful for analysing *in situ* measurements on board satellites when they are close to the source.

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